Christ Church Grammar School

2021
TEST 3

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	2 minutes
Working time for this section:	25 minutes
Marks available:	23 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

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3. Answer all questions.

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6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 1 **(6 marks)**

(a) Express $\frac{4x+3}{9x-6x^2}$ in the format $\frac{a}{3x}+\frac{b}{3-2x}$ where $a, b \in \mathbb{R}$. (3 marks)

$$\frac{4x+3}{9x-6x^2} = \frac{a}{3x} + \frac{b}{3-2x}$$

$\Rightarrow \quad 4x+3 = a(3-2x) + b(3x)$ ✓ equates numerators correctly

$x=0 \quad 3 = 3a \quad \therefore a = 1$ ✓ solves to determine 'a'

$x=1.5 \quad 9 = 4.5b \quad \therefore b = 2$ ✓ solves to determine 'b' and writes equation in correct form

$$\frac{4x+3}{9x-6x^2} = \frac{1}{3x} + \frac{2}{3-2x}.$$

Note: can achieve marks 2 and 3 if step 1 is wrong but you can clearly see working.

(b) Hence, determine $\int \frac{4x+3}{9x-6x^2}\,dx$. (3 marks)

$$\int \frac{4x+3}{9x-6x^2}\,dx = \int \left(\frac{1}{3x} + \frac{2}{3-2x}\right) dx$$ ✓ recognises to use part (a) for $\int$.

$$= \frac{1}{3}\ln|x| + -\ln|3-2x| + C$$

✓ 1 correct term ✓ 2nd correct term with + C

See next page

Question 2 (7 marks)

(a) Determine $\int \left(\frac{\pi \sin x - \pi}{\sqrt{x+\cos(x)}}\right) dx$ (4 marks)

uses correct substitution

let $u = x + \cos x$

$\frac{du}{dx} = 1 - \sin x$

$dx = \frac{du}{1-\sin x}$

$= -\pi \int \frac{-\sin x + 1}{u^{1/2}} \cdot \frac{du}{1-\sin x}$

$= -\pi \int u^{-1/2}\, du$ ✓ correctly ∫ in terms of u

$= -\pi \frac{u^{1/2}}{1/2} + c$ ✓ integrates correctly

$= -2\pi u^{1/2} + c$

$= -2\pi \sqrt{x+\cos x} + c$

✓ subst back in original function

or

$-\pi \int \frac{1-\sin x}{(x+\cos x)^{1/2}}\, dx$

recognises derivative of $(x+\cos x) = 1-\sin x$ ✓

✓ Factors out negative

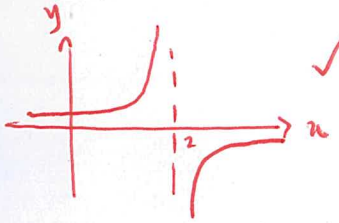
$= -\pi \int (1-\sin x)(x+\cos x)^{-1}\, dx$

integrates correctly ✓ $= -\pi \frac{(x+\cos x)^{1/2}}{1/2} + c$

$= -2\pi (x+\cos x)^{1/2} + c$

correct answer ✓ $= -2\pi \sqrt{x+\cos x} + c$

(b) Evaluate $\int_1^3 \left(\frac{x}{4-x^2}\right) dx$ (3 marks)



✓ draws diagram

$x \neq -2$ ∴ ~~integral~~ area does not converge

state area doesn't converge

∴ no solution ✓ final statement

or if let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$.

If $x = 1$, $u = 3$

$x = 3$ $u = -5$

$\int_3^{-5} \frac{x}{u} \frac{du}{-2x}$

$= -\frac{1}{2} \int_3^{-5} \frac{1}{u}\, du$

$= -\frac{1}{2} \ln|u| \Big]_3^{-5}$

$= \left(-\frac{1}{2}\ln|-5| - \left(-\frac{1}{2}\ln|3|\right)\right)$

$= -\frac{1}{2}\ln 5 + \frac{1}{2}\ln 3$

✓ substs & change limits

$= \frac{1}{2}\ln\left(\frac{3}{5}\right)$

$= \ln\left(\frac{3}{5}\right)^{1/2}$ ✓ integrates & simplifies correctly

or $\ln\sqrt{\frac{3}{5}}$

however, no solution due to $x \neq -2$. ✓ Final statement

∴ doesn't converge

or $\int_1^3 \frac{x}{4-x^2}\, dx$

$= -\frac{1}{2}\ln|4-x^2|\Big]_1^3$

$= \left(-\frac{1}{2}\ln(4-9)\right) - \left(-\frac{1}{2}\ln|4-1|\right)$

$= -\frac{1}{2}\ln 5 + \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left(\frac{3}{5}\right)$

$= \ln\left(\frac{3}{5}\right)^{1/2}$ or $\ln\left(\sqrt{\frac{3}{5}}\right)$

however, no solution as $x \neq -2$.

∴ doesn't converge

See next page

Question 3 **(5 marks)**

Determine the following definite integral using an appropriate substitution.

$$\int_0^1 12x(2x-1)^4\,dx$$

let $u = 2x - 1$

✓ $\frac{du}{dx} = 2$ correct subst. and dx in terms of du

$dx = \frac{du}{2}$

$x = \frac{u+1}{2}$

✓ If $x = 0$ $u = -1$

$x = 1$ $u = 1$

correct new limits.

$\therefore \int_{-1}^{1} 12\left(\frac{u}{2} + \frac{1}{2}\right) u^4 \frac{du}{2}$ correctly changes all variables

$= 3\int_{-1}^{1} (u^5 + u^4)\,du$ ✓

$= 3\left[\frac{u^6}{6} + \frac{u^5}{5}\right]_{-1}^{1}$ ✓ integrates correctly

$= 3\left[\left(\frac{1}{6} + \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{5}\right)\right]$

$= 3\left[\frac{2}{5}\right]$

$= \frac{6}{5}$ ✓ correct answer.

Question 4 **(5 marks)**

Determine

$$\int_0^{\frac{\pi}{4}} (cos(x) + 2\,sin(x))^2\,dx$$

Note: could also integrate $\int_0^{\pi/4}(\cos^2x + \sin^2x + 4\sin x\cos x + 3\sin^2x)\,dx$
$= \int_0^{\pi/4}(1 + 2\sin 2x + 3\sin^2x\,dx$

$= \int_0^{\pi/4} (\cos x + 2\sin x)(\cos x + 2\sin x)\,dx$

$= \int_0^{\pi/4} (\cos^2x + 4\sin x\cos x + 4\sin^2x)\,dx$ ✓ expands and simplifies ( ) correctly

$= \int_0^{\pi/4} (\cos^2x + 2\sin 2x + 4 - 4\cos^2x)\,dx$

$= \int_0^{\pi/4} (2\sin 2x + 4 - 3\cos^2x)\,dx$

$\cos 2x = 2\cos^2x - 1$

$\frac{\cos 2x + 1}{2} = \cos^2x$

$= \int_0^{\pi/4} (2\sin 2x + 4 - 3(\frac{\cos 2x}{2} + \frac{1}{2}))\,dx$ ✓ simplifies and uses trig subst. correctly

$= \int_0^{\pi/4} (2\sin 2x + \frac{5}{2} - \frac{3}{2}\cos 2x)\,dx$

$= \left[-\frac{1}{2}\cdot 2\cos 2x + \frac{5x}{2} - \frac{3}{2}\cdot\frac{1}{2}\sin 2x\right]_0^{\pi/4}$ ✓ integrates correctly

$= \left(-\cos 2x + \frac{5x}{2} - \frac{3}{4}\sin 2x\right)_0^{\pi/4}$

$= \left(-\cos\frac{\pi}{2} + \frac{5\pi/4}{2} - \frac{3}{4}\sin 2\tfrac{\pi}{2}\right) - \left(-\cos 0 + 0 - \frac{3}{4}\sin 0\right)$ ✓ subst. in lower & upper bounds

$= \left(0 + \frac{5\pi}{8} - \frac{3}{4}\right) - (-1)$

$= \frac{5\pi}{8} + \frac{1}{4}$ ✓ correct final answer.

End of questions

Additional working space

Question number: _____

Additional working space

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TEST 3

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS.

Teacher's name _____

Time and marks available for this section

Reading time:	2 minutes
Working time for this section:	20 minutes
Marks available:	18 marks

Materials required/recommended for this section

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Special items: drawing instruments, templates, and up to three calculators approved for use in this assessment

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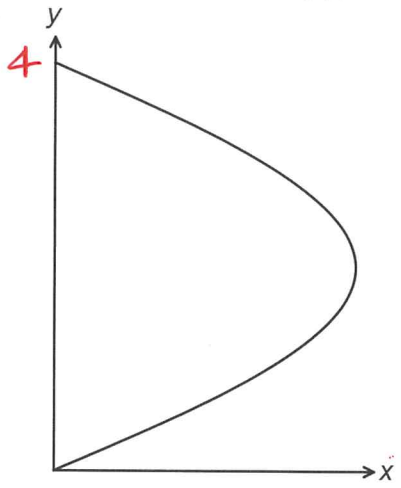
See next page

Question 5 **(5 marks)**

A section of the graph of the curve $x = sin\left(\frac{\pi y}{4}\right)$ in the first quadrant is sketched below.



(a) Determine the area of the region bounded by the curve $x = sin\left(\frac{\pi y}{4}\right)$ and the y – axis. (3 marks)

$\int_0^4 \sin\left(\frac{\pi y}{4}\right) dy$ ✓ determines upper bound.

$= \left[-\frac{4}{\pi}\cos\left(\frac{\pi y}{4}\right)\right]_0^4$ ✓ correctly integrates (or sets up integral from above)

$\simeq \frac{8}{\pi}$ $(2.54647... \simeq 2.5)$ ✓ correct value (accept exact or appropriate rounding).

(b) Determine the volume of the solid if it is rotated around the y – axis for one revolution. (2 marks)

$V = \pi \int_0^4 x^2 dy$

$= \pi \int_0^4 \left(\sin\left(\frac{\pi y}{4}\right)\right)^2 dy$ ✓ sets up correct integral

$= 2\pi$ units^3. ✓ correct volume (accept exact value or appropriate rounding)

or $6.283185 \simeq 6.3$ units^3

Note: correct answer only award 2 marks.

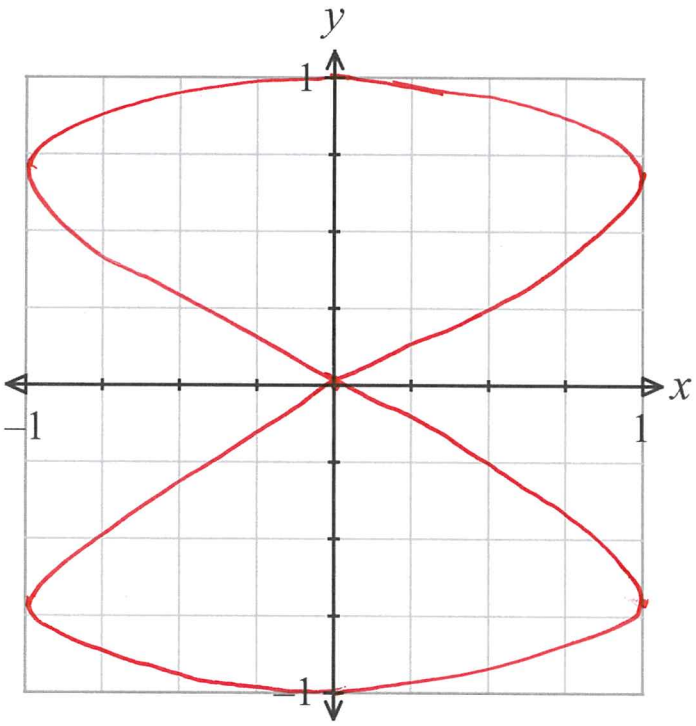
See next page

Question 6 **(9 marks)**

A particle's position vector, $\boldsymbol{r}(t)$ in metres, at any time t seconds is given by the equation

$$\boldsymbol{r}(t) = \begin{pmatrix} \sin(2t) \\ \sin t \end{pmatrix}$$

(a) Plot the path of the particle on the axes shown below. (3 marks)



✓ shows intersections at (0,1), (0,−1) and (0,0)

✓ symmetrical

✓ correct shape

(b) Determine the Cartesian equation of the path in the form $x^2 = f(y)$. (2 marks)

$\underset{\sim}{r}(t) = \begin{pmatrix} \sin 2t \\ \sin t \end{pmatrix}$

$\therefore x = \sin 2t$

$= 2\sin t \cos t$

$x^2 = 4\sin^2 t \cos^2 t$

$y = \sin t$

$y^2 = \sin^2 t$

✓ squares both components

and $\cos^2 t = 1 - \sin^2 t$

$\therefore x^2 = 4y^2(1 - y^2)$

✓ correct answer

Note: correct answer only award 2 marks.

See next page

Question 6 continued

(c) Determine the speed of the particle when it reaches the point where $x = -0.5$ for the second time. (4 marks)

$\underset{\sim}{v}(t) = \begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix}$

'x' $\Rightarrow -0.5 = \sin 2t$ ✓ solves for $x = -0.5$

$t = \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$

$\therefore t = \frac{11\pi}{12}$ (second time) ✓ determines correct 't' value

Speed $= \sqrt{\left(2\cos 2\left(\frac{11\pi}{12}\right)\right)^2 + \left(\cos\left(\frac{11\pi}{12}\right)\right)^2}$ ✓ states speed equation (or velocity)

$= 1.98$ m/s. ✓ correctly calculate the speed (based on their equation)

accept appropriate rounding.

See next page

Question 7 **(4 marks)**

Determine, the solution(s) for the following systems of equations. In each case, interpret the systems of equations geometrically.

(a) $8x + y + z = 15$
$2x + y - z = 3$
$x - y + 2z = 3$ (2 marks)

$z = t$
$y = 5t - 1$
$x = -\frac{t}{3} + 2$

✓ determines solution (can use any variable)

Infinitely many solutions where the planes are not parallel so they represent 3 planes that intersect in a line

✓ correct geometric explanation. Must state planes are not parallel and intersect in a line

(b) $x + y = z - 2$
$x - y + z = 1$
$x + z = y + 3$ (2 marks)

$\Rightarrow$
$x + y - z = -2$
$x - y + z = 1$
~~$x + z = 3$~~
$x - y + z = 3$

} // planes

No solution ✓ correct solution

Two planes are parallel and they are then cut by the third plane. ✓ correct geometric explanation.

End of questions

Additional working space

Question number: _____

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